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Test 2 Name ______________________

Show all necessary work for full credit. You may use your calculator.

1. **Set up** the intial-value problems for the following scenarios (give the differential equation and initial conditions).

3 A. An RL circuit has a resistance of 10 ohms, an inductance of 1.5 henries, an applied emf of 9 volts, and an initial current of 6 amperes. Write the differential equation in terms of the charge.

$$1.5\frac{d^2q}{dt^2} + 10\frac{dq}{dt} = 9 \qquad \frac{d^2q}{dt^2} + \frac{20}{3}\frac{dq}{dt} = 6$$

$$i(0) = 6$$

4 B. A 20lb weight is attached to a spring having a spring constant of 5lb/ft. The mass is started in motion from the equilibrium position with an initial velocity of 2 ft/sec in the upward direction and with an external force of F(t) = 5sin(t). Assume that the damping force due to air resistance is 1 times the instantaneous velocity.

$$5/8\frac{d^2x}{dt^2} + \frac{dx}{dt} + 5x = 5\sin(t)$$

$$x(0) = 0 \qquad x'(0) = -2$$

4 C. An 8 lb weight is attached to a spring whereupon the spring is stretched 2 feet and allowed to come to rest. The weight is set into motion from rest by displacing the spring 6 inches above its equilibrium position with an initial velocity of 3 ft/sec in the downward direction. Assume the surrounding medium offers a negligible resistance.

$$\frac{1}{4}\frac{d^2x}{dt^2} + 4x = 0$$

$$x(0) = -\frac{1}{2}$$

$$x'(0) = 3$$

4 D) An RCL circuit connected in series with a resistance of 4 ohms, a capacitor of 1/26 fard, and an inductance of ½ henry has no applied voltage. The initial charge is 1/10 coulomb and the initial current is zero.

$$1/2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 26q = 0$$

$$q(0) = \frac{1}{10}$$

$$q'(0) = i(0) = 0$$

2. A) Solve y" + 4y' + 5y = 0.

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$$\lambda^2 + 4\lambda + 5 = 0$$

$$\lambda = -2 \pm i$$

$$y(t) = e^{-2t}(c_1\cos t + c_2\sin t)$$

4 B) Solve y" + 7y' + 6y = 0.

$$\lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda+6)(\lambda+1) = 0$$

$$\lambda = -6 \quad \lambda = -1$$

$$y(t) = c_1 e^{-6t} + c_2 e^{-t}$$

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3. Determine the form of the particular solution, y_p, to $L(y) = \phi(x)$ for the given $\phi(x)$ if the solution to the homogeneous equation $L(y) = 0$ is $y_h = c_1 + c_2e^{4x} + c_3e^{-x}$.

3 A) $\phi(x) = 3x^2 + 5$

$$y_P = (Ax^2 + Bx + C)x = Ax^3 + Bx^2 + Cx$$

3 B) $\phi(x) = 7\sin(5x)$

$$y_P = A\sin(5x) + B\cos(5x)$$

3 C) $\phi(x) = 6e^{5x} + 2e^{-x} + 3$

$$y_P = Ae^{5x} + Bxe^{-x} + Cx$$

3 D) $\phi(x) = x^2\cos(x)$

$$y_P = (Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin(x)$$

4. A complete set of roots for the characteristic equation of an 4^{th}-order homogeneous differential equation in $y(x)$ with real numbers as coefficients is $0, 3, \pm 2i$.

4 A) Determine the general solution of the differential equation.

$$y_H(x) = C_1 + C_2e^{3x} + C_3\sin(2x) + C_4\cos(2x)$$

4 B) Determine the differential equation associated with the roots given.

$$\lambda(\lambda-3)(\lambda^2+4) = \lambda(\lambda^3 - 3\lambda^2 + 4\lambda - 12) = \lambda^4 - 3\lambda^3 + 4\lambda^2 - 12\lambda$$

$$y^{(4)} - 3y''' + 4y'' - 12y' = 0$$

4 5. Solve $y^{(4)} + 10y''' = 0$.

$$\lambda^4 + 10\lambda^3 = 0$$

$$\lambda^3(\lambda + 10) = 0$$

$$\lambda = 0 \quad \lambda = -10$$

$$y(x) = C_1 + C_2x + C_3x^2 + C_4e^{-10x}$$

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(6) 6. Solve $2y'' + 12y = 4e^{3x}$; $y(0) = 1$ and $y'(0) = 0$ using the method of undetermined coefficients.

$2\lambda^2 + 12 = 0$

$\lambda^2 + 6 = 0$

$\lambda = \pm\sqrt{6}i$

$y_H = c_1 \cos(\sqrt{6}x) + c_2 \sin(\sqrt{6}x)$

$y_P = Ae^{3x}$

$y_P' = 3Ae^{3x}$

$y_P'' = 9Ae^{3x}$

$18Ae^{3x} + 12Ae^{3x} = 4e^{3x}$

$30A = 4$

$A = \frac{2}{15}$

$y(x) = c_1 \cos(\sqrt{6}x) + c_2 \sin(\sqrt{6}x) + \frac{2}{15}e^{3x}$

$y(0)=1: \quad 1 = c_1 + \frac{2}{15} \quad c_1 = \frac{13}{15}$

$y'(x) = -c_1\sqrt{6}\sin(\sqrt{6}x) + \sqrt{6}c_2\cos(\sqrt{6}x) + \frac{2}{5}e^{3x}$

$y'(0)=0: \quad 0 = \sqrt{6}c_2 + \frac{2}{5} \quad c_2 = \frac{-2}{5\sqrt{6}}$

$y(x) = \frac{13}{15}\cos(\sqrt{6}x) - \frac{2}{5\sqrt{6}}\sin(\sqrt{6}x) + \frac{2}{15}e^{3x}$

(6) 7. Solve $2y'' - 2y' - 4y = xe^{2x}$ using variation of parameters.

$y'' - y' - 2y = \frac{1}{2}xe^{2x}$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2 \quad \lambda = -1$

$y_H(x) = c_1 e^{2x} + c_2 e^{-x}$

$W = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} = -e^{x} - 2e^{x} = -3e^{x}$

$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{2}xe^{2x} & -e^{-x} \end{vmatrix} = -\frac{1}{2}xe^{x}$

$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{1}{2}xe^{2x} \end{vmatrix} = \frac{1}{2}xe^{4x}$

$u_1' = \frac{-\frac{1}{2}xe^{x}}{-3e^{x}} = \frac{1}{6}x$

$u_1 = \frac{1}{12}x^2$

$u_2' = \frac{\frac{1}{2}xe^{4x}}{-3e^{x}} = -\frac{1}{6}xe^{3x}$

$u_2 = \frac{-1}{18}xe^{3x} + \frac{1}{54}e^{3x}$

$y(x) = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{12}x^2 e^{2x} + \left(-\frac{1}{18}xe^{3x} + \frac{1}{54}e^{3x}\right)e^{-x}$

$= c_1 e^{2x} + c_2 e^{-x} + \frac{1}{12}x^2 e^{2x} - \frac{1}{18}xe^{2x}$

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Test 2 Take-home part   Name ____________________

This should be an individual effort. You may use your book, calculator and notes to complete this. Using the internet, tutors, friends, etc is considered cheating. Please do all problems neatly and in order on your own paper. Show all necessary work for full credit. This is due at the beginning of class on October 26. No late papers will be accepted.

7 1. Solve the IVP using the method of undetermined coefficients: $y''' - 3y'' + 3y' - y = e^x - 5x + 12$;

$y(0) = 0$; $y'(0) = 0$; $y''(0) = 0$.

7 2. Solve the DE $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$ using variation of parameters subject to the initial conditions $y(0) = 2$ and $y'(0) = 1$.

3. A mass of weighing 32 lbs is suspended from a spring whose spring constant is 9 lb/ft. The mass is initially released from a point 1 foot above equilibrium with an upward velocity of $\sqrt{3}$ ft/s.

5 A) Find the equation of motion of the spring.

3 B) Write the equation in the form $x(t) = A \sin(\omega t + \phi)$

3 C) Find the times at which the mass passes through the equilibrium position.

4. A mass weighing 12 pounds stretches a spring 2 feet. The mass is attached to a device that offers a damping force numerically equal to β ($\beta > 0$) times the instantaneous velocity. Determine the values of the damping constant β so that the subsequent motion is:

2 A) overdamped

2 B) underdamped

2 C) critically damped

5. An electromotive force of $E(t) = \sin(5t)$ is applied to an LR-series circuit in which the inductance is 0.5 henries and the resistance is 10 ohms. Find the current $i(t)$ if $i(0) = 0$. 6

6. An RCL circuit connected in series with a resistance of 8 ohms, a condenser of capacitance of 0.001 farad, and an inductance of 1 henry has an applied emf $E(t) = 110$ volts. Assuming no initial current and no initial charge on the capacitor, find the charge on the capacitor. 7

1. $y''' - 3y'' + 3y' - y = e^x - 5x + 12$

$(\lambda - 1)^3 = 0 \qquad \lambda = 1$

$y_H = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

$y_p = Ax^3 e^x + Bx + C$

$y'_p = Ax^3 e^x + 3Ax^2 e^x + B$

$y''_p = Ax^3 e^x + 3Ax^2 e^x + 3Ax^2 e^x + 6Axe^x = Ax^3 e^x + 6Ax^2 e^x + 6Axe^x$

$y'''_p = Ax^3 e^x + 9Ax^2 e^x + 18Axe^x + 6Ae^x$

$\cancel{Ax^3e^x} + \cancel{9Ax^2e^x} + \cancel{18Axe^x} + \underline{6Ae^x} - \cancel{3Ax^3e^x} - \cancel{18Ax^2e^x} - \cancel{18Axe^x} +$
$\cancel{3Ax^3e^x} + \cancel{9Ax^2e^x} + 3B - \cancel{Ax^3e^x} - Bx - C = e^x - 5x + 12$

$6A = 1 \qquad -B = -5 \qquad 3B - C = 12$

$A = \frac{1}{6} \qquad B = 5 \qquad C = 3$

$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + \frac{1}{6} x^3 e^x + 5x + 3$

$y(0) = 0 \qquad 0 = c_1 + 3 \qquad c_1 = -3$

$y'(0) = 0 \qquad y' = c_1 e^x + c_2 (xe^x + e^x) + c_3 (x^2 e^x + 2xe^x) + \frac{1}{2} x^2 e^x$
$+ \frac{1}{6} x^3 e^x + 5$

$0 = -3 + c_2 + 5 \qquad c_2 = -2$

$y''(0) = 0 \qquad y'' = -3e^x - 2(xe^x + 2e^x) + c_3 (x^2 e^x + 4xe^x + 2e^x) +$
$\frac{1}{2} x^2 e^x + xe^x + \frac{1}{2} x^2 e^x + \frac{1}{6} x^3 e^x$

$0 = -3 - 4 + 2c_3 \qquad c_3 = 7/2$

$y = -3e^x - 2xe^x + \frac{7}{2} x^2 e^x + \frac{1}{6} x^3 e^x + 5x + 3$

2. $4y'' - 4y' + y = e^{x/2}\sqrt{1-x^2}$ $\quad y(0)=2 \quad y'(0)=1$

$(2\lambda-1)^2=0 \quad \lambda=\frac{1}{2}$

$y_H = c_1 e^{1/2x} + c_2 x e^{1/2x}$

$$W = \begin{vmatrix} e^{1/2x} & xe^{1/2x} \\ \frac{1}{2}e^{1/2x} & \frac{1}{2}xe^{1/2x}+e^{1/2x} \end{vmatrix} = \frac{1}{2}xe^x + e^x - \frac{1}{2}xe^x = e^x$$

$$W_1 = \begin{vmatrix} 0 & xe^{1/2x} \\ \frac{1}{4}e^{x/2}\sqrt{1-x^2} & \frac{1}{2}xe^{1/2x}+e^{1/2x} \end{vmatrix} = -\frac{1}{4}xe^x\sqrt{1-x^2}$$

$$W_2 = \begin{vmatrix} e^{1/2x} & 0 \\ \frac{1}{2}e^{1/2x} & \frac{1}{4}e^{x/2}\sqrt{1-x^2} \end{vmatrix} = \frac{1}{4}e^x\sqrt{1-x^2}$$

$u_1' = -\frac{1}{4}x\sqrt{1-x^2} \qquad u_1 = -\frac{1}{4}\int x\sqrt{1-x^2}\,dx = \frac{1}{12}(1-x^2)^{3/2}$

$u_2' = \frac{1}{4}\sqrt{1-x^2} \qquad u_2 = \frac{1}{8}\left(\sin^{-1}(x) + x\sqrt{1-x^2}\right)$

$y(x) = c_1 e^{1/2x} + c_2 x e^{1/2x} + \frac{1}{12}(1-x^2)^{3/2}e^{1/2x} + \frac{1}{8}\left(\sin^{-1}(x)+x\sqrt{1-x^2}\right)xe^{1/2x}$

$2 = c_1 + \frac{1}{12} \qquad c_1 = 23/12 \qquad c_2 = 0$

$y = 23/12\, e^{1/2x} + \frac{1}{12}(1-x^2)^{3/2}e^{1/2x} + \frac{1}{8}\left(\sin^{-1}(x)+x\sqrt{1-x^2}\right)xe^{1/2x}$

3. 32 lbs $\quad k = 9\,\text{lb}/\text{ft} \quad x(0) = -1 \quad x'(0) = -\sqrt{3}\ \text{ft}/\text{sec}$

(5) $m = \frac{32}{32} = 1$ slug

A. $\frac{d^2x}{dt^2} + 9x = 0$

$\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

$x = c_1 \cos(3t) + c_2 \sin(3t)$

$-1 = c_1$

$x' = 3c_1 \sin(3t) + 3c_2 \cos(3t)$

$-\sqrt{3} = 3c_2$

$-\frac{\sqrt{3}}{3} = c_2$

$x = -\cos(3t) - \frac{\sqrt{3}}{3}\sin(3t)$

B. $A = \sqrt{(-1)^2 + \left(-\frac{\sqrt{3}}{3}\right)^2} = \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

(3) $\tan\phi = \frac{-1}{-\frac{1}{\sqrt{3}}} = \sqrt{3}$

$\phi = \frac{4\pi}{3}$

$x(t) = \frac{2}{\sqrt{3}} \sin\left(3t + \frac{4\pi}{3}\right)$

C. $0 = \frac{2}{\sqrt{3}} \sin\left(3t + \frac{4\pi}{3}\right)$

(3) $3t + 4\pi/3 = k\pi$

$t = \frac{k\pi - \frac{4\pi}{3}}{3} = \frac{k}{3}\pi - \frac{4}{9}\pi$

4. $m = \frac{12}{32} = \frac{3}{8}$ slugs $\qquad 12 = k(2)$

$k = 6$

$$\frac{3}{8}\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 6x = 0$$

$$\beta^2 - 4\left(\frac{3}{8}\right)(6) = \beta^2 - 9 > 0$$

$$\beta^2 > 9$$

A. $|\beta| > 3$ over

C. $|\beta| = 3$ critically

B. $|\beta| < 3$ under

(6) 5. $.5\frac{d^2q}{dt^2} + 10\frac{dq}{dt} = \sin(5t) \qquad i(0) = 0$

$$\frac{d^2q}{dt^2} + 20\frac{dq}{dt} = 2\sin(5t)$$

$$\lambda^2 + 20\lambda = 0$$
$$\lambda(\lambda + 20) = 0$$
$$\lambda = 0 \quad \lambda = -20$$

$$q_H(t) = c_1 + c_2e^{-20t}$$

$$q_p(t) = A\sin(5t) + B\cos(5t)$$
$$q'_p(t) = 5A\cos(5t) - 5B\sin(5t)$$
$$q''_p(t) = -25A\sin(5t) - 25B\cos(5t)$$

$$-25A\sin(5t) - 25B\cos(5t) + 100A\cos(5t) - 100B\sin(5t) = 2\sin(5t)$$

$(-25A - 100B = 2)4 \qquad -100A - 400B = 8 \qquad B = \frac{-8}{425} \quad A = \frac{-2}{425}$

$100A - 25B = 0 \qquad 100A - 25B = 0$

$-425B = 8$

$$q(t) = c_1 + c_2e^{-20t} - \frac{2}{425}\sin(5t) - \frac{8}{425}\cos(5t)$$

$$i(t) = -20c_2e^{-20t} - \frac{2}{85}\cos(5t) + \frac{8}{85}\sin(5t)$$

$0 = -20c_2 - \frac{2}{85} \qquad c_2 = \frac{\frac{2}{85}}{-20} = \frac{-1}{850}$

$$i(t) = \frac{2}{85}e^{-20t} - \frac{2}{85}\cos(5t) + \frac{8}{85}\sin(5t).$$

10.
(7)

$$\frac{d^2q}{dt^2} + 8\frac{dq}{dt} + \frac{1}{.001}q = 110 \qquad i(0)=0 \quad q(0)=0$$

$$\frac{d^2q}{dt^2} + 8\frac{dq}{dt} + 1000q = 110$$

$$\lambda^2 + 8\lambda + 1000 = 0$$

$$\lambda = -4 \pm 2\sqrt{246}\, i$$

$$q_H(t) = e^{-4t}\left(c_1 \cos\left(2\sqrt{246}\,t\right) + c_2 \sin\left(2\sqrt{246}\,t\right)\right)$$

$$q_P(t) = A$$

$$1000A = 110$$

$$A = \frac{11}{100}$$

$$q(t) = e^{-4t}\left(c_1 \cos\left(2\sqrt{246}\,t\right) + c_2 \sin\left(2\sqrt{246}\,t\right)\right) + \frac{11}{100}$$

$q(0)=0$: $0 = c_1 + \frac{11}{100} \qquad c_1 = \frac{-11}{100}$

$$i(t) = e^{-4t}\left(-2\sqrt{246}\,c_1 \sin\left(2\sqrt{246}\,t\right) + 2\sqrt{246}\,c_2 \cos\left(2\sqrt{246}\,t\right)\right)$$
$$-4e^{-4t}\left(c_1 \cos\left(2\sqrt{246}\,t\right) + c_2 \sin\left(2\sqrt{246}\,t\right)\right)$$

$i(0)=0$ $\qquad 0 = 2\sqrt{246}\,c_2 - 4c_1$

$$0 = 2\sqrt{246}\,c_2 - 4\left(-\frac{11}{100}\right)$$

$$-\frac{11}{25} = 2\sqrt{246}\,c_2$$

$$\frac{-11}{50\sqrt{246}} = c_2$$

$$q(t) = e^{-4t}\left[\frac{-11}{100}\cos\left(2\sqrt{246}\,t\right) - \frac{11}{50\sqrt{246}}\sin\left(2\sqrt{246}\,t\right)\right] + \frac{11}{100}$$